Many-body reduced fidelity susceptibility in Lipkin-Meshkov-Glick model

Jian Ma,[1,2] Xiaoguang Wang,[2,*] and Shi-Jian Gu[1,†]

[1]*Department of Physics and ITP, The Chinese University of Hong Kong, Hong Kong 999077, China*

[2]*Department of Physics, Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, People's Republic of China*

(Received 2 July 2009; published 27 August 2009)

We study the reduced fidelity susceptibility χ_r for an M-body subsystem of an N-body Lipkin-Meshkov-Glick model with $\tau=M/N$ fixed. The reduced fidelity susceptibility can be viewed as the response of subsystem to a certain parameter. In noncritical region, the inner correlation of the system is weak, and χ_r behaves similar with the global fidelity susceptibility χ_g, the ratio $\eta=\chi_r/\chi_g$ depends on τ but not on N. However, at the critical point, the inner correlation tends to be divergent, and we find χ_r approaches χ_g with increasing the N. It is interesting to note that, $\eta=1$ in the thermodynamic limit, which means the susceptibilities of the local and global system are the same. Finally, we make numerical computations, and they are in perfect agreement with the analytical predictions.

DOI: 10.1103/PhysRevE.80.021124 PACS number(s): 64.60.−i, 03.67.−a, 64.70.Tg, 03.65.Ud

I. INTRODUCTION

Quantum phase transition (QPT) [1], which occurs at absolutely zero temperature, is driven purely by quantum fluctuations. It was studied conventionally by Landau paradigm with order parameter in the frame of statistics and condensed-matter physics. Recently, two quantum-information [2] concepts, entanglement [3–13], and fidelity [14–28] have been investigated extensively in QPTs and are recognized to be effective and powerful in detecting the critical point. The former measures quantum correlations between partitions, while the latter measures the distance in quantum state space. Therefore, their success in characterizing QPTs is understood by regarding the universality of the critical behaviors itself, that is, the divergent of the correlation and the dramatic change in the ground-state structure. Furthermore, as the fidelity depends computationally on an arbitrarily small change in the driving parameter, Zarnardi *et al.* suggested the Riemannian metric tensor [18], while You *et al.* suggested the fidelity susceptibility [19]; both focus on the leading term of the fidelity. In the following, we mainly consider the fidelity susceptibility (FS).

Until now, most efforts have been devoted to the study of the global ground-state fidelity susceptibility (GFS), denoted by χ_g, which reflects the susceptibility of the system in response to the change in certain driving parameter. In this work, we study the responses of a subsystem, for which we study its FS, the so-called reduced fidelity susceptibility (RFS), denoted by χ_r. Some special cases have been studied in Refs. [20,26–29], where the subsystems are only one body or two body, while in this paper we will study an arbitrary M-body subsystem. The motivation for the investigation of RFS is clear in physics. First, it reveals information about the change in the inner structure for a system that undergoes QPT. Second, as the existence of interactions and correlations, a general quantum system is not the simple addition of its different parts, especially in the critical region, where the entanglement entropy is divergent [5,6,10]. Therefore it is significant to investigate the behavior of the RFS, as well as the effects of entanglement on it, in both critical and noncritical regions. And our study can be viewed as a connection between the FS and the entanglement entropy.

To study this question, we consider an N-body Lipkin-Meshkov-Glick model (LMG) [30] model, and study the RFS for its M-body subsystem. As $0\le\chi_r\le\chi_g$ [28], we consider a more useful quantity, $\eta=\chi_r/\chi_g$, and thus $\eta\in[0,1]$. We find that, the behaviors of the RFS, as well as η, are quite different in noncritical and critical regions. In noncritical region, the entanglement entropy is saturated by a finite upper bound, and the inner correlation is small, thus the RFS behaves similar with the GFS, and the ratio η depends on $\tau=M/N$ but not N. However, at the critical point, the entanglement entropy tends to be divergent with the increase in system size, and the inner correlations are very strong. Then we find the RFS approaches GFS with the increase in N, and $\eta=1$ in the thermodynamic limit for $\tau\neq0$. These can be understood by considering the divergent of correlation in second-order QPTs, which is reflected by the entanglement entropy.

This paper is organized as follows: in Sec. II, we present the LMG model and give a brief review of the GFS studied in Ref. [27]. Then in Sec. III, we derive the RFS in the thermodynamic limit and obtain its divergent form in the vicinity of the critical point. Then we perform some numerical computations, and the results are in perfect agreement with our analytical prediction.

II. LMG MODEL AND GLOBAL FIDELITY SUSCEPTIBILITY

The LMG model was originally introduced in nuclear physics and has found applications in a broad range of other topics: statistical mechanics of quantum spin system [31], Bose-Einstein condensates [32], or magnetic molecules such as Mn_{12} acetate [33], as well as quantum entanglement [34], and quantum fidelity [27,28]. It is an exactly solvable [35,36] many-body interacting quantum system as well as one of the simplest to show a quantum transition in the regime of strong

*xgwang@zimp.zju.edu.cn

†sjgu@phy.cuhk.edu.hk

coupling. The quantum phase transition of this model can be described by the symmetry broken mechanism, the two phases are associated with either collective or single-particle behavior. The Hamiltonian of the LMG model reads

$$H = -\frac{\lambda}{N}(S_x^2 + \gamma S_y^2) - hS_z, \tag{1}$$

where $S_\alpha = \sum_{i=1}^{N}\sigma_\alpha^i/2$ ($\alpha = x, y, z$) are the collective spin operators; σ_α^i are the Pauli matrices; N is the total spin number; γ is the anisotropic parameter. λ and h are the spin-spin interaction strength and the effective external field, respectively. Here, we focus on the ferromagnetic case ($\lambda > 0$), and without loss of generality, we set $\lambda = 1$ and $0 \leq \gamma \leq 1$. As the spectrum is invariant under the transform $h \leftrightarrow -h$, we only consider $h \geq 0$. This system undergoes a second-order QPT at $h=1$, between a symmetric (polarized, $h > 1$) phase and a broken (collective, $h < 1$) phase, which is well described by a mean-field approach [37]. The classical state is fully polarized in the field direction ($\langle\sigma_z^i\rangle = 1$) for $h > 1$, and is twofold degenerate with $\langle\sigma_z^i\rangle = h$ for $h < 1$.

Before deriving the RFS, we give a brief review of the GFS and its application in the LMG model [27]. Just as the specific heat is a thermal response function, GFS can be viewed as a response function for the ground state with respect to the external driving parameter. As the ground state is a pure state, the fidelity between two ground states is just their overlap

$$F = |\langle\psi(h)|\psi(h+\delta h)\rangle|, \tag{2}$$

where $|\psi(h)\rangle$ is the ground state and h is the driving parameter. If δh is small, the above fidelity can be expanded to the second order of δh, $F = 1 - \chi(\delta h)^2/2$, the first-order term is zero due to the normalization of states [19]. Therefore, the GFS can be evaluated with standard perturbation method, as presented in Ref. [27]; the authors employed the Holstein-Primakoff transform and derived the GFS for both phases in the thermodynamic limit,

$$\chi_g(h,\gamma) = \begin{cases} \dfrac{N}{4\sqrt{(1-h^2)(1-\gamma)}} + \dfrac{h^2(h^2-\gamma)^2}{32(1-\gamma)^2(1-h^2)^2}, & \text{for} \quad 0 \leq h < 1, \\ \dfrac{(1-\gamma)^2}{32(h-\gamma)^2(h-1)^2}, & \text{for} \quad h \geq 1. \end{cases} \tag{3}$$

It has been found that, when $h < 1$, the GFS increases with N and can be viewed as an extensive quantity. However, when $h > 1$ the GFS is saturated with an upper bound, i.e., it is intensive.

III. REDUCED FIDELITY SUSCEPTIBILITY

A. Thermodynamic limit

Now we give some basic formulas for fidelity and its susceptibility. In fact, there is no difference between GFS and RFS in mathematical definitions, they are both leading terms of the fidelity. However, since the subsystem is represented by a mixed state, we should use a more general form fidelity, the Uhlmann fidelity [43],

$$F(\rho,\tilde{\rho}) \equiv \mathrm{tr}\sqrt{\rho^{1/2}\tilde{\rho}\rho^{1/2}}, \tag{4}$$

where $\rho \equiv \rho(h)$ and $\tilde{\rho} \equiv \rho(h+\delta h)$ with a certain parameter h. When ρ and $\tilde{\rho}$ are pure states, it returns to Eq. (2). If δh tends to zero, the two states are close in parameter space, and their Bures distance [42] is,

$$ds_B^2 = 2[1 - F(\rho,\tilde{\rho})]. \tag{5}$$

In the basis of ρ, denoted by $\{|\psi_i\rangle\}$, the Bures distance can be written as [44]

$$ds_B^2 = \frac{1}{4}\sum_{n=1}^{N}\frac{dp_n^2}{p_n} + \frac{1}{2}\sum_{n\neq m}^{N}\frac{(p_n-p_m)^2}{p_n+p_m}|\langle\psi_n|d\psi_m\rangle|^2, \tag{6}$$

where p_i are the eigenvalues of ρ and N is the dimension of ρ. As FS is the leading term of fidelity, i.e., $F = 1 - \chi(\delta h)^2/2$, we can get FS for h immediately,

$$\chi(h) = \frac{1}{4}\sum_{n=1}^{N}\frac{(\partial_h p_n)^2}{p_n} + \frac{1}{2}\sum_{n\neq m}^{N}\frac{(p_n-p_m)^2}{p_n+p_m}|\langle\psi_n|\partial_h\psi_m\rangle|^2, \tag{7}$$

where $\partial_h := \partial/\partial h$. In our study, ρ and $\tilde{\rho}$ are just the reduced density matrices for ground states.

In the following, the N-body system is divided into two parts, A and B with size M and $N-M$, respectively. Without loss of generality, we will study the RFS for subsystem A, the reduced density matrix is ρ_A. This study would give a connection between the RFS and the entanglement entropy [10]. As we know, the entanglement reflects the correlation among inner partitions, and our study will reveal the effects of these correlations on RFS, especially at the critical point.

Now we introduce the total spin operators for the two subsystems, $S_\alpha^{A,B} = \sum_{i\in A,B}\sigma_\alpha^i/2$. To describe quantum fluctuations, it is convenient to use the Holstein-Primakoff representation of the spin operators [38], and the first step is to rotate the z axis along the semiclassical magnetization

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \tilde{S}_x \\ \tilde{S}_y \\ \tilde{S}_z \end{pmatrix}. \quad (8)$$

As presented in Ref. [37], $\theta_0=0$ for $h>1$ so that $\mathbf{S}=\tilde{\mathbf{S}}$, and $\theta_0=\arccos h$ for $h\le 1$. The Holstein-Primakoff representation is then applied to the rotated spin operators

$$\tilde{S}_z^A = M/2 - a^\dagger a,$$

$$\tilde{S}_-^A = \sqrt{M}a^\dagger\sqrt{1-a^\dagger a/M} = (\tilde{S}_+^A)^\dagger,$$

$$\tilde{S}_z^B = (N-M)/2 - b^\dagger b,$$

$$\tilde{S}_-^B = \sqrt{N-M}b^\dagger\sqrt{1-b^\dagger b/(N-M)} = (\tilde{S}_+^B)^\dagger, \quad (9)$$

where $a(a^\dagger)$ and $b(b^\dagger)$ are bosonic creation and annihilation operators for subsystem A and B, respectively, and $S_\pm^{A,B}=S_x^{A,B}\pm iS_y^{A,B}$. After this transform, the LMG Hamiltonian is mapped onto a system of two interacting bosonic modes a and b. For fixed $\tau=M/N$, the Hamiltonian can be expanded in $1/N$. Up to the order $(1/N)^0$, one gets $H=NH^{(-1)}+H^{(0)}+O(1/N)$ with $H^{(-1)}=(m^2-1-2h)/4$, where $m=\cos\theta_0$, and

$$H^{(0)} = -\frac{1+\gamma}{4} + \mathbf{A}^\dagger\mathbf{V}A^T + \frac{1}{2}[\mathbf{A}^\dagger\mathbf{W}(\mathbf{A}^\dagger)^T + \text{h.c.}], \quad (10)$$

where $\mathbf{A}=(a,b)$, and

$$\mathbf{V} = \frac{2hm+2-3m^2-\gamma}{2}\mathbb{I}$$

$$\mathbf{W} = \frac{\gamma-m^2}{2}\begin{pmatrix} \tau & \sqrt{\tau(1-\tau)} \\ \sqrt{\tau(1-\tau)} & 1-\tau \end{pmatrix}, \quad (11)$$

where $\mathbb{I}$ is a 2×2 identity matrix; $m=h$ in broken phase and $m=1$ in symmetric phase. The bosonic Hamiltonian can be diagonalized by Bogoliubov transform, and we will see that it is useful in deriving the reduced density matrix. As shown in Refs. [39–41], the reduced density matrix for eigenstates of a quadratic form can always be written as $\rho_A=e^{-\mathcal{H}}$ with

$$\mathcal{H} = \kappa_0 + \kappa_1 a^\dagger a + \kappa_2(a^{\dagger 2}+a^2). \quad (12)$$

κ_i $(i=0,1,2)$ can be determined by using [10]

$$\text{tr}\,\rho_A = 1, \text{tr}(\rho_A a^\dagger a) = \langle a^\dagger a\rangle \quad \text{and} \quad \text{tr}(\rho_A a^{\dagger 2}) = \langle a^{\dagger 2}\rangle, \quad (13)$$

where $\langle\Omega\rangle=\langle\psi_g|\Omega|\psi_g\rangle$, $|\psi_g\rangle$ is the ground state, Then we can diagonalize ρ_A by Bogoliubov transform. However, in this paper we will adopt another method to diagonalize ρ_A, as shown in Ref. [11], ρ_A is written in the bosonic coherent-state representation

$$\langle\phi|\rho_A|\phi'\rangle = K\exp\left[\frac{1}{4}(\phi^*+\phi')\frac{G^{++}-1}{G^{++}+1}(\phi^*+\phi')\right] \times \exp\left[\frac{1}{4}(\phi^*-\phi')\frac{G^{--}+1}{G^{--}-1}(\phi^*-\phi')\right],$$

where $a|\phi\rangle=\phi|\phi\rangle$; $K=\sqrt{(1+G^{++})(1-G^{--})}$ is determined by the normalization of ρ_A; G^{++} and G^{--} are Green's functions defined as

$$G^{++} = \langle(a^\dagger+a)^2\rangle,$$

$$G^{--} = \langle(a^\dagger-a)^2\rangle. \quad (14)$$

Then ρ^A can be diagonalized by the following Bogoliubov transform:

$$g = \cosh\varphi a + \sinh\varphi a^\dagger = \frac{P+Q}{2}a + \frac{P-Q}{2}a^\dagger, \quad (15)$$

with $PQ=1$, $PG^{++}=\mu Q$, and $QG^{--}=-\mu P$. The Green's functions can be obtained by diagonalizing the bosonic represented by Hamiltonian (10),

$$G^{++} = 1 + (1/\alpha - 1)\tau,$$

$$G^{--} = (1-\alpha)\tau - 1, \quad (16)$$

where

$$\alpha = \begin{cases} \sqrt{\dfrac{h-1}{h-\gamma}} & \text{for} \quad h\ge 1, \\ \sqrt{\dfrac{1-h^2}{1-\gamma}} & \text{for} \quad 0\le h<1. \end{cases} \quad (17)$$

The diagonalized ρ^A reads

$$\rho^A = \frac{2}{\mu+1}e^{-\varepsilon g^\dagger g}, \quad (18)$$

where the pseudoenergy $\varepsilon=\ln[(\mu+1)/(\mu-1)]$ with $\mu=\alpha^{-1/2}\sqrt{[\tau\alpha+(1-\tau)][\tau+\alpha(1-\tau)]}$.

Now we can derive the RFS (7), of which the first term involves only the eigenvalues of ρ_A, and the second term involves both the eigenvalues and the eigenvectors. The eigenvectors of ρ_A is the number state $|n\rangle: g^\dagger g|n\rangle=n|n\rangle$, and the term $|\langle\psi_n|\partial_h\psi_m\rangle|^2=|\langle n|\partial_h m\rangle|^2$ can be calculated as

$$|\langle n|\partial_h m\rangle|^2 = \frac{|\langle n|\partial_h(g^\dagger g)|m\rangle|^2}{(m-n)^2}. \quad (19)$$

Therefore, we can write the RFS explicitly,

$$\chi_r(h,\gamma,\tau) = \frac{(\partial_h\mu)^2}{4(\mu^2-1)} + \frac{(\mu\partial_h\varphi)^2}{\mu^2+1} + \frac{N\tau}{4\mu}(\partial_h\theta_0\exp\varphi)^2, \quad (20)$$

where $\varphi=\text{arctanh}[(\mu-G^{++})/(\mu+G^{++})]$, $\theta_0=\arccos h$ for $h\le 1$ and $\theta_0\equiv 0$ for $h>1$. Thus the last term of the above expression only takes effect in the broken phase. We emphasize that, in the broken phase $h<1$, we should perform a rotation (8) at first.

We can also express the RFS as

$$\chi_r(h,\gamma,\tau)=\begin{cases}\chi+\dfrac{N\tau}{4G^{++}(1-h^2)} & \text{for} \quad 0\le h<1,\\ \chi & \text{for} \quad h\ge 1,\end{cases} \tag{21}$$

where

$$\chi=\frac{(\partial_h\mu)^2}{4(\mu^2-1)}+\frac{\mu^2}{4(\mu^2+1)}\left[\partial_h \ln\left(-\frac{\mu}{G^{++}}\right)\right]^2. \tag{22}$$

In the vicinity of the critical point, the RFS diverges as

$$\chi_r/N\propto(1-h)^{-1/2},\text{for} \quad 0\le h<1, \tag{23}$$

$$\chi_r\propto(1-h)^{-2},\text{for} \quad h\ge 1, \tag{24}$$

and this is the same with χ_g. Additionally, we show the entanglement entropy $\mathcal{E}=-\mathrm{tr}(\rho\ln\rho)$ that was derived in [10,11],

$$\mathcal{E}=\frac{\mu+1}{2}\ln\frac{\mu+1}{2}-\frac{\mu-1}{2}\ln\frac{\mu-1}{2}+x\ln 2, \tag{25}$$

where $x=1$ when $h<1$ and $x=0$ when $h>1$, the ln 2 term comes from the twofold degeneracy of the ground state in the broken phase, and this degeneracy is lifted for finite N. The entanglement entropy diverges as $(1/4)\ln|h-1|$ around the critical point, and is nearly independent with N in noncritical region.

B. Finite-size cases

To perform numerical computations, we should derive the reduced density matrix for ρ_A in finite-size case. The LMG model is of high symmetry in interaction, and the ground state which is the superposition of the Dick states lies in the $J=N/2$ section

$$|\psi_g\rangle=\sum_{m=0}^{N}C_m|J,-J+m\rangle, \tag{26}$$

where C_m is the coefficient to be determined numerically. We hope to write $|J,-J+m\rangle$ in the form of $|J_A,m_A\rangle|J_B,m_B\rangle$, where $J_A=M/2$ and $J_B=(N-M)/2$ correspond to the two local systems. Since $|J,-J+m\rangle=\sqrt{(2J-m)!/(2J)!m!}(S_+)^m|J,-J\rangle$, and the ladder operator $S_+=S_+^A+S_-^B$. Then the ground state is

$$|\psi_g\rangle=\sum_{m=0}^{N}\sum_{p=0}^{2J_A}C_m\sqrt{\mathrm{H}(p;2J,2J_A,m)}|J_A,-J_A+p\rangle\otimes|J_B,-J_B+m-p\rangle, \tag{27}$$

where

$$\mathrm{H}(p;2j,2j_1,m)=\frac{\binom{2j_1}{p}\binom{2j_2}{m-p}}{\binom{2j}{m}} \tag{28}$$

is the so-called Hypergeometric distribution function. And the matrix element of ρ_A is

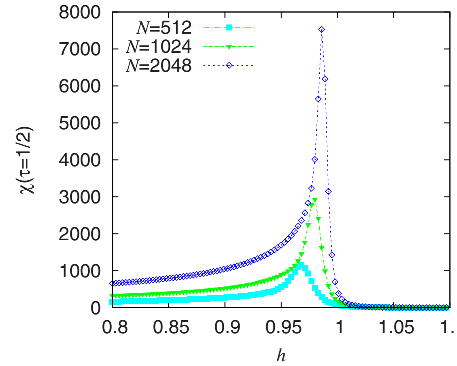

FIG. 1. (Color online) RFS as a function of h at $\gamma=1/2$ and $\tau=1/2$. The peaks approach the critical point and become sharper and sharper with the increase in N.

$$(\rho_A)_{p,q}=\sum_{m=0}^{N}C_mC^*_{q+m-p}\sqrt{\mathrm{H}(p;2J,2J_A,m)}\times\sqrt{\mathrm{H}(q;2J,2J_A,q+m-p)}. \tag{29}$$

By using the exact diagonalization method, the RFS as a function of h for fixed τ is computed and shown in Fig. 1. As one can see, the peaks of the RFS approach the critical point and become sharper and sharper with the increase in N. The RFS in the symmetric phase ($h>1$) has an upper bound; however, in the broken phase ($h<1$) the RFS increases with the total spin number N. Thus we address that the RFS is extensive in the broken phase, in which the LMG model is of collective behavior while it is intensive in the symmetric phase, in which the LMG model behaves like a single particle. This is similar with the GFS [27].

As $0\le\chi_r\le\chi_g$, we present a more significative quantity $\eta(\tau,h)\equiv\chi_r(h,\gamma,\tau)/\chi_g(h,\gamma)$ and focus on its properties in critical and noncritical regions. With Eqs. (3) and (20), we find that in the thermodynamic limit

$$\lim_{h\to 1}\eta(\tau,h)=1, \tag{30}$$

for any nonvanishing τ. To verify our prediction, we show the analytical and numerical results in Fig. 2. As one can see, at the critical point, the RFS approaches the global one, i.e., η tends to one, and at the same time, the entanglement entropy, i.e., the inner correlation between subsystems A and B, is divergent with the increase in N. When h is away from the critical region, the inner correlation decreases dramatically, and then η depends on τ but not the total system size N as shown in Fig. 3.

As demonstrated in Ref. [28], when there are no correlations between partitions of a system, for example, an N-body system represented by a product state that reads

$$|\psi(h)\rangle=\bigotimes_{i=1}^{N}|\phi_i(h)\rangle, \tag{31}$$

if we denote a one-body reduced fidelity as F_r, the relation between the global and the reduced fidelities is

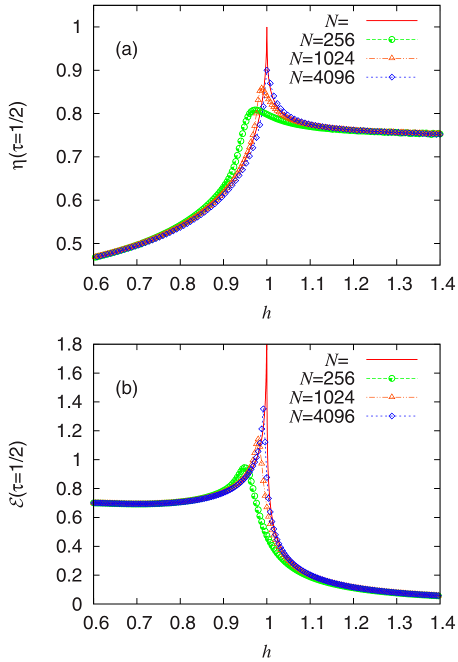

FIG. 2. (Color online) A comparison between (a) η and (b) $\mathcal{E}$ as a function of h at $\gamma=1/2$, $\tau=1/2$ for various system sizes. At the critical point, η tends to 1 while $\mathcal{E}$ is divergent.

$$F_g(h,\delta) = \prod_{i=1}^{n} F_r^i(h,\delta), \tag{32}$$

and thus we have $\chi_g=\Sigma_{i=1}^{N}\chi_r^i$; moreover, if the system is of translation symmetry, we have $\chi_g=N\chi_r$. If there is entanglement between partitions, we have no such results, especially in the critical point, the entanglement is divergent, and then $\chi_g/\chi_r=1$ in the thermodynamic limit. This is some kind of effect of the inner correlations on the susceptibility of the system states. However, we address that our results are based on a high-dimension model, in which the interaction is infinite range. We think it deserved to study the RFS for a contiguous block in a low-dimension model, for example, the XY model in which the interaction is just between neighboring sites. Thus the correlation between a block and its complementary part takes effect only at the boundary, and the results for η may be different.

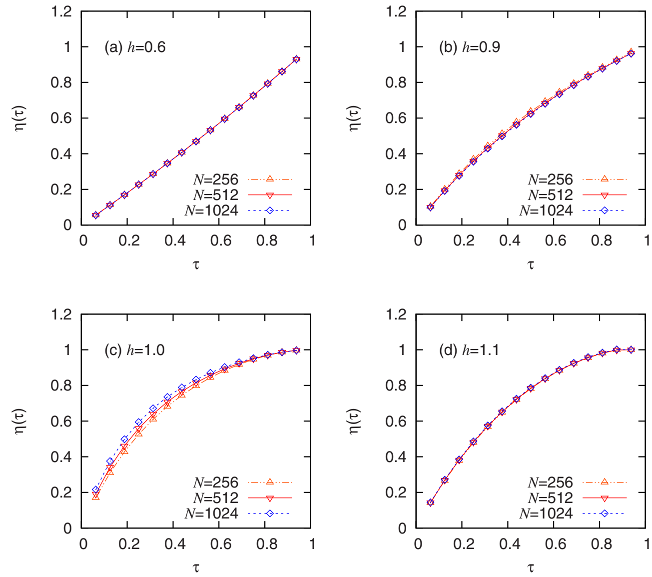

FIG. 3. (Color online) η as a function of τ with $\gamma=1/2$, at (a) $h=0.6$, (b) 0.9, (c) 1.0, and (d) 1.1. We see that η is independent of N when h is away from the critical region.

IV. CONCLUSION

In conclusion, we derive the RFS analytically in the thermodynamic limit for a fixed τ. To analyze the effects of the inner correlations on the RFS, we study the ratio $\eta=\chi_r/\chi_g$ combined with the entanglement entropy in both critical and noncritical regions. Our results give a clear picture for understanding the effects of correlations on the response. In the critical region, with the increase in N, the entanglement entropy tends to be divergent and η approaches 1, while in the thermodynamic limit, $\eta\equiv 1$ for $\tau\neq 0$. This indicates that the sensitivity of the subsystem is equal to the global one at the critical point, where the correlation is very strong. In noncritical region, the RFS behaves similarly with the GFS, and η depends on τ but not on N.

[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999); M. Vojta, Rep. Prog. Phys. **66**, 2069 (2003).
[2] M. A. Nilesen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[3] X. Wang, Phys. Rev. A **64**, 012313 (2001).
[4] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
[5] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
[6] J. I. Latorre, E. Rico, and G. Vidal, Quantum Inf. Comput. **4**, 048 (2004).
[7] J. Vidal, G. Palacios, and C. Aslangul, Phys. Rev. A **70**, 062304 (2004).
[8] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004).
[9] J. I. Latorre, R. Orús, E. Rico, and J. Vidal, Phys. Rev. A **71**, 064101 (2005).
[10] T. Barthel, S. Dusuel, and J. Vidal, Phys. Rev. Lett. **97**, 220402 (2006).
[11] T. Barthel, M. C. Chung, and U. Schollwöck, Phys. Rev. A **74**, 022329 (2006).
[12] R. Orús, S. Dusuel, and Julien Vidal, Phys. Rev. Lett. **101**, 025701 (2008).
[13] H. T. Cui, Phys. Rev. A **77**, 052105 (2008).
[14] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
[15] P. Zanardi and N. Paunkovic, Phys. Rev. E **74**, 031123 (2006).

[16] P. Buonsante and A. Vezzani, Phys. Rev. Lett. **98**, 110601 (2007).
[17] P. Zanardi, M. Cozzini, and P. Giorda, J. Stat. Mech.: Theory Exp. (2007) L02002; M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B **75**, 014439 (2007); M. Cozzini, R. Ionicioiu, and P. Zanardi, *ibid.* **76**, 104420 (2007).
[18] P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007).
[19] W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).
[20] H. Q. Zhou, and J. P. Barjaktarevic, J. Phys. A: Math. Theor. **41**, 412001 (2008); H. Zhou, J. Zhao, and B. Li; J. Phys. A: Math. Theor. **41**, 492002 (2008); H. Zhou, e-print arXiv:0704.2945.
[21] L. Campos Venuti and P. Zanardi, Phys. Rev. Lett. **99**, 095701 (2007).
[22] S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B **77**, 245109 (2008).
[23] S. Chen, L. Wang, S. J. Gu, and Y. Wang, Phys. Rev. E **76**, 061108 (2007).
[24] W. Q. Ning, S. J. Gu, Y. G. Chen, C. Q. Wu, and H. Q. Lin, J. Phys.: Condens. Matter **20**, 235236 (2008).
[25] M. F. Yang, Phys. Rev. B **76**, 180403(R) (2007); Y. C. Tzeng and M. F. Yang, Phys. Rev. A **77**, 012311 (2008).
[26] N. Paunkovic, P. D. Sacramento, P. Nogueira, V. R. Vieira, and V. K. Dugaev, Phys. Rev. A **77**, 052302 (2008).
[27] H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, Phys. Rev. E **78**, 032103 (2008).
[28] J. Ma, L. Xu, H. N. Xiong, and X. Wang, Phys. Rev. E **78**, 051126 (2008).
[29] E. Eriksson and H. Johannesson, Phys. Rev. A **79**, 060301(R) (2009).
[30] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. **62**, 188 (1965); A. J. Glick, H. J. Lipkin, and N. Meshkov, *ibid.* **62**, 211 (1965).
[31] R. Botet, R. Jullien, and P. Pfeuty, Phys. Rev. Lett. **49**, 478 (1982).
[32] J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, Phys. Rev. A **57**, 1208 (1998).
[33] D. A. Garanin, X. Martinez Hídalgo, and E. M. Chudnovsky, Phys. Rev. B **57**, 13639 (1998).
[34] J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A **69**, 022107 (2004).
[35] F. Pan and J. P. Draayer, Phys. Lett. B **451**, 1 (1999).
[36] J. Links, H. Q. Zhou, R. H. McKenzie, and M. D. Gould, J. Phys. A **36**, R63 (2003).
[37] S. Dusuel and J. Vidal, Phys. Rev. B **71**, 224420 (2005).
[38] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[39] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).
[40] I. Peschel and M. C. Chung, J. Phys. A **32**, 8419 (1999).
[41] I. Peschel, J. Phys. A **36**, L205 (2003).
[42] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969).
[43] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976); **24**, 229 (1986).
[44] H. J. Sommers and K. Zyczkowski, J. Phys. A **36**, 10083 (2003).